Probabilistic Aspects of Computer Science: TD1
Discrete-time Renewal Processes and First Markov Chains

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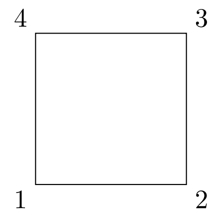
Exercise 1. We consider here a discrete-time renewal process with delay probability distribution $(f_i)_{i \in \mathbb{N}^*}$ having a finite support, i.e., there is a finite set of delays $i_1 < \cdots < i_k = N$ such that: $f_i \neq 0$ iff. there exists $\ell \in \{1, \ldots, k\}$ such that $i = i_\ell$. We want to prove in this special case that if the distribution f is aperiodic with mean value μ then $\lim_{n\to\infty} u_n = \mu^{-1}$.

1. Solve the case $k = 1$.
2. We now suppose that $k > 1$. Show that if the sequence (u_n) admits a limit, then it is μ^{-1}.
3. Recall the recurrence relation verified by the sequence (u_n) and show that 1 is a simple root of the associated characteristic polynomial.
4. Show that the only root of this characteristic polynomial with radius greater or equal to 1 is 1 itself. Conclude.

Exercise 2. Consider a person walking on the following square.



The person starts at intersection 1. Then, at every intersection, the person tosses a fair coin and if the coin turns up heads then the person moves anti-clockwise, otherwise the person moves clockwise.

1. Describe the situation with a finite discrete-time Markov chain, with transition matrix $\mathbf{P}$. Give the transition graph of the Markov chain. What is the initial distribution π_0?
2. Compute the distribution $\pi_n = \pi_0 \mathbf{P}^n$ for all natural numbers n. Does this Markov chain admit a steady-state distribution?
3. Now, consider the same example with different rules. Instead of tossing one faire coin, a person tosses two fair coins. If the first coin turns up heads then the person decides to stay in the position it was before; otherwise the person tosses another coin. If the second coin turns up heads, the person moves anti-clockwise, otherwise the person moves clockwise. Give the new transition matrix $\mathbf{Q}$ and the transition graph. Compute the new distribution π_n and study the existence of a steady-state distribution.

*Taken from last year exercises by Benjamin Monmege.

Exercise 3. From the following processes, which are Markov chains? For those that are, supply the transition matrix. For those that are not, give a Markov chain that is equivalent to this process, if there exists one.

1. A dice is rolled repeatedly. Let X_n be the largest number shown up to the nth roll.
2. A dice is rolled repeatedly. At time r, let C_r be the time since the most recent six.
3. A dice is rolled repeatedly. At time r, let B_r be the time until the next six.
4. Consider a discrete event system $X_0, X_1, X_2, \ldots$ with state space S. The process is governed by two matrices $\mathbf{P}$ and $\mathbf{Q}$. If k is even, the values $\mathbf{P}[i,j]$ give the probability of going from state i to state j on the step from X_k to X_{k+1}. Likewise, if k is odd, the values $\mathbf{Q}[i,j]$ give the probability of going from state i to state j on the step from X_k to X_{k+1}.
5. Let (X_n) be a Markov chain. Consider the process $(X_n, X_{n+1})_{n\geq 0}$.

Exercise 4. We give a very simplified version of the internet assumed by the PageRank algorithm employed by search algorithms (for example, by Google). The algorithm assumes that the internet consists of some webpages which have hyperlinks to other webpages. The person browsing the internet decides (probabilistically) whether to click on one of these links or visit a new page by entering it in the address bar.

Assuming that there are N webpages in the world named $p_1, \ldots, p_N$, the PageRank algorithm creates a Markov chain M with N states $\{p_1, \ldots, p_N\}$. If the webpage p_i has links to every other page then we let $\mathbf{P}[i,j] = \frac{1}{N-1}$ for every $j \neq i$. If p_i has links to $N' < N-1$ webpages then $\mathbf{P}[i,j] = \frac{0.85}{N'}$ if p_i has a link to page p_j $(j \neq i)$ and $\mathbf{P}[i,j] = \frac{0.15}{N-N'-1}$ if p_i does not have a link to page p_j $(j \neq i)$. The initial distribution of M assigns probability $\frac{1}{N}$ to each of the states.

1. Is the Markov chain aperiodic?
2. Is the Markov chain irreducible?

Exercise 5.

1. Let X_n be the number of heads obtained after n independent tosses of a (possibly unfair) coin. Show that, for any $k \geq 2$,
$$\lim_{n\to\infty} \mathbf{Pr}(X_n \text{ is divisible by } k) = \frac{1}{k}\,.$$
Give a bound on the rate of convergence. (*Hint: Model the problem with a Markov chain, and consider the sequence* $w_n = \max \pi_{kn} - \min \pi_{kn}$.)
2. Let $\mathbf{M}$ be a square matrix of size k. We suppose that $\mathbf{M}$ is regular, i.e., there exists $p > 0$ such that for every i, j, $\mathbf{M}^p[i,j] > 0$. We also assume the existence of $v > 0$ (i.e., for all i, $v[i] > 0$) such that $v\,\mathbf{M} = v$. Prove that for every $v_0 \geq 0$, the sequence defined by $v_{n+1} = v_n\,\mathbf{M}$ admits a limit which is moreover proportional to v.
3. Show that if P is a regular stochastic matrix, then the associated Markov chain admits a unique steady-state distribution.

Exercise 6. Consider a sequence of independent, fair gambling games between two players. In each round, a player wins a coin with probability 1/2 or loses a coin with probability 1/2. The state of the system at time n is the number of coins won by player 1. The initial state is 0. Assume that there are numbers ℓ_1 and ℓ_2 such that player i cannot lose more than ℓ_i coins, and thus the game ends when it reaches one of the two states $-\ell_1$ or ℓ_2. At this point, one of the gamblers is ruined.

1. Describe the Markov chain associated with this game and find the probability that player 1 wins ℓ_2 coins before losing ℓ_1 coins.
2. Classify the states and find again the probability that player 1 wins ℓ_2 coins before losing ℓ_1 coins, using this classification.
3. Solve the problem in the case of an unfair game with a probability $p > 1/2$ for player 1 to lose one coin on each round.