Probabilistic Aspects of Computer Science: TD2
Markov chains in the long run

Emile Contal*
http://econtal.perso.math.cnrs.fr/teaching

October 3, 2013

Exercise 1. We study again the same exercise than last week, but with simpler tools.

1. Let X_n be the number of heads obtained after n independent tosses of a (possibly unfair) coin. Show that, for any $k \geq 2$,
$$\lim_{n\to\infty} \mathbf{Pr}(X_n \text{ is divisible by } k) = \frac{1}{k}$$
2. Solve the problem when X_n represents the sum of n independent rolls of a dice.

Exercise 2. Exhibit a Markov chain which has null recurrent states (different from the one studied in course).

Exercise 3. Show that if a Markov chain has two steady-state distributions, then it has an infinite number of steady-state distributions.

Exercise 4 (Move-to-front heuristic)**.** Suppose that we are given $n \geq 2$ records $R_1, R_2, \dots, R_n$. The records are kept in some order. The cost of accessing the jth record in the order is j. Thus, if we had four records ordered as R_2, R_4, R_3, R_1, then the cost of accessing R_4 would be 2 and the cost of accessing R_1 would be 4.

Suppose further that, at each step, record R_j is accessed with probability p_j, with each step being independent of other steps.

1. If we knew the values of the p_j in advance, what is the best choice to order the records?

   We suppose now that we do not know the p_j in advance and we use a *move-to-front* heuristic: at each step, put the record that was accessed at the front of the list. We assume that moving the record can be done with no cost and that all other records remain in the same order. For example, if the order was R_2, R_4, R_3, R_1 before R_3 was accessed, then the order at the next step would be R_3, R_2, R_4, R_1.

2. Describe this problem with a Markov chain and find the steady-state distribution of this chain, if it exists.

3. Let X_k be the cost for accessing the kth requested record. Show that
$$\lim_{k\to\infty} \mathbf{E}(X_k) = \frac{1}{2} + \sum_{i,j} \frac{p_i p_j}{p_i + p_j}\,.$$

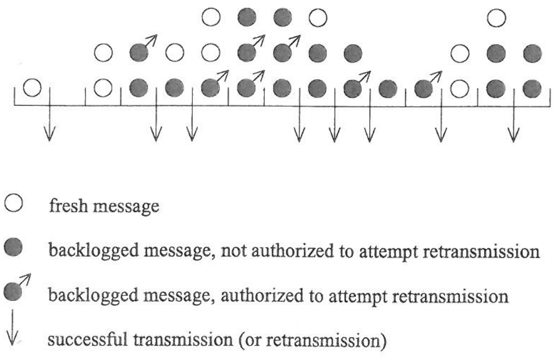
Exercise 5 (ALOHA)**.** A typical situation in a multiple-access satellite communications system is the following. Users, each one identified with a message, contend for access to a single-channel satellite communications link for the purpose of transmitting messages. Two or more messages in the air at the same time jam each other, and are not successfully transmitted. The users are somehow able to detect a collision of this sort and will try to retransmit later the message involved in a collision. The difficulty in such communications systems resides mainly in the absence of cooperation among users, who are all unaware of the intention to transmit of competing users. The slotted ALOHA protocol imposes on the users the following rules:

*Taken from last year exercises by Benjamin Monmege.

(i) Transmissions and retransmissions of messages can start only at equally spaced moments; the interval between two consecutive (re-)transmission times is called a *slot*; the duration of a slot is always larger than that of any message.

(ii) All *backlogged* messages, i.e., those messages having already tried unsuccessfully – maybe more than once – to get through the link, require retransmission independently of one another with probability $\nu \in (0,1)$ at each slot. This is the so-called *Bernoulli retransmission policy*.

(iii) The *fresh messages* – those presenting themselves for the first time – immediately attempt to get through.



Let X_n be the number of backlogged messages at the beginning of slot n.

1. Supposing there are $X_n = k$ backlogged messages, express the probability $b_i(k)$ that i among them attempt to retransmit in slot n as a function of i, k and ν.

2. Let A_n be the number of fresh requests for transmission in slot n. Supposing that the sequence $\{A_n\}_{n\geq 0}$ is assumed i.i.d. with the distribution $\mathbf{Pr}(A_n = j) = a_j$, give a condition over the sequence $(a_j)_{j\geq 0}$ for the sequence $\{X_n\}_{n\geq 0}$ to be described by an irreducible Markov chain.

3. Show that this chain is not positive recurrent: we say that the system using the Bernoulli retransmission policy is *not stable*.

   In the following, we admit and use the *Pakes' Lemma*:

   > Let $\{X_n\}_{n\geq 0}$ be an irreducible Markov chain with states $\mathbb{N}$, such that for all $i, n \geq 0$
   > $$\mathbf{E}(X_{n+1} \mid X_n = i) < \infty$$
   > and
   > $$\limsup_{i\to\infty} \mathbf{E}(X_{n+1} - X_n \mid X_n = i) < 0\,.$$
   > Then the Markov chain is positive recurrent.

4. We now consider a retransmission policy stabilizing ALOHA. Assume the retransmission probability ν now depends on the number k of backlogged messages. Express the expectations appearing in Pakes' Lemma as a function of $\nu(k)$, a_0, a_1 and $\lambda \overset{\text{def}}{=} \mathbf{E}(A_n) = \sum_{i=1}^{\infty} i a_i$ (the so-called *traffic intensity*, supposed finite from now on).

5. Using Pakes' Lemma, design a $\nu(i)$ and find a sufficient condition over λ, a_0 and a_1 for stability of this protocol.

6. Supposing that the arrivals $\{A_i\}$ follow a Poisson distribution of parameter λ

   $$a_i = e^{-\lambda}\frac{\lambda^i}{i!}\,,$$

   find a condition over λ for the ALOHA protocol to be stable.