Probabilistic Aspects of Computer Science: TD4
Continuous-time Markov chains

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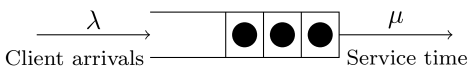
Exercise 1. Let $\lambda\mu > 0$ and let X be a Markov chain on $\{1, 2\}$ with infinitesimal generator

$$\mathbf{Q} = \begin{pmatrix} -\mu & \mu \\ \lambda & -\lambda \end{pmatrix}$$

1. Write down the forward equations and solve them for finding the transition probabilities $\pi_{ij}(t)$, $i, j \in \{1, 2\}$.
2. Calculate $\mathbf{Q}^n$ and hence find $\sum_{n=0}^{\infty} \frac{t^n}{n!}\mathbf{Q}^n$. Compare your answer with that to previous question.
3. Solve the global balance equation $\mathbf{u} \cdot \mathbf{Q} = \mathbf{0}$ in order to find the steady-state distribution $\mathbf{u}$. Verify that $\pi_{ij}(t) \to \mathbf{u}_j$ as $t \to \infty$.

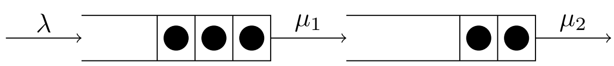
Exercise 2 (Waiting queue)**.** We study the behavior of a client queue in front of a service as represented below:



Interarrival times of client are independent identically distributed, and their common distribution is an exponential one with rate λ (we say that the arrivals follow a *Poisson process*). The service time has also an exponential distribution with rate μ. One client is served at a time, and arrivals of clients are independent of service times.

1. Describe this model with a continuous-time Markov chain.
2. Find a necessary and sufficient condition over the *traffic intensity* $\rho = \lambda/\mu$ for the existence of a steady-state distribution for this Markov chain, that you will then compute.

Exercise 3 (Tandem queue)**.** We consider the case of a tandem queue. Clients arrive to a first queue with interarrival times i.i.d., with an exponential distribution of rate λ, and with service time independent and exponentially distributed with parameter μ_1. After completing service at this first queue, the clients proceed immediately to a second queue, where service times are also independent and exponentially distributed with parameter μ_2:



Find a necessary and sufficient condition over the traffic intensities $\rho_1 = \lambda/\mu_1$ and $\rho_2 = \lambda/\mu_2$ for the existence of an equilibrium of this system, that you will then describe.

*Taken from last year exercises by Benjamin Monmege.

Exercise 4 (Open networks)**.** We consider an open network of interconnected queues. There are K stations, and station $k \in \{1, \dots, K\}$ has a unique service system, such that service times are independent and exponentially distributed with parameter μ_k. There are two types of clients queuing at a given station: (1) those which are fed-back, that is, who have received service in another or the same station and are rerouted to the given station for more service, and (2) those who enter the network for the first time. We suppose that exogenous arrivals in station k are such that interarrival times follow an exponential distribution of parameter λ_k. Moreover, the re-routing process is made with respect to a *routing matrix* $\mathbf{R}$: after a client completes a service in station k, he tosses a $(K+1)$-faced dice with probabilities $r_{k,1}, \dots, r_{k,K}, r_k = 1 - \sum_{j=1}^{K} r_{k,j}$ with the effect that the client is sent to station j with probability $r_{k,j}$ or leaves the system with probability r_k. We suppose that the successive tosses of the routing dice of all stations are independent and independent of the exogenous arrival processes and of all the service times.

1. Describe the continuous-time Markov chain associated with this system, and give its infinitesimal generator $\mathbf{Q}$.

2. Under which condition, this Markov chain is irreducible? Interpret these conditions in terms of *exogenous suppliance* and *outlets*. In the following, we suppose that the Markov chain always fulfills these conditions.

3. Show that the following system of equations, that we call *traffic equations*, has a unique solution that you will express with respect to the vector λ of exogenous rates and the matrix $\mathbf{P}$:

$$u_k = \lambda_k + \sum_{j=1}^{K} u_j r_{jk}, \text{ for } k \in \{1, \dots, K\}\,.$$

   Supposing that the Markov chain is ergodic, show that this is the average number of customers entering station k at the steady state.

4. We define the traffic intensity of station k as $\rho_k = u_k/\mu_k$. Show that if $\rho_k < 1$ for all $k \in \{1, \dots, K\}$, then the system admits a unique steady-state distribution that you will determine. (*Hint: You can use, after proving it, the fact that if π is a strictly positive distribution, and $\tilde{\mathbf{Q}}$ the generator defined by $\pi_{\mathbf{n}}\tilde{q}_{\mathbf{n},\mathbf{n}'} = \pi_{\mathbf{n}'}q_{\mathbf{n}',\mathbf{n}}$ then,*

$$\forall \mathbf{n} \quad \sum_{\mathbf{n}' \neq \mathbf{n}} \tilde{q}_{\mathbf{n},\mathbf{n}'} = -q_{\mathbf{n},\mathbf{n}}$$

   is a sufficient condition for π to be the steady-state distribution.)

Exercise 5 (Closed networks)**.** We consider a closed network of interconnected queues, i.e., an open network as defined in previous exercice verifying that for every station k, $\lambda_k = r_k = 0$. In that case, there is no inlet and no outlet, and therefore the number of customers in the networks remains constant, and we shall call it N. We follow the same outline than in the previous exercise. In particular, we suppose that the underlying Markov chain is irreducible.

1. Give the traffic equations of this network and show that there is a unique solution with norm 1.

2. Show that the network has always a steady-state distribution, that you will express using the solution of traffic equations, and a normalizing constant $G(N, K)$.

3. In practice, it is impossible to compute with a brute-force summation this normalizing constant. Show that we can still compute it with a dynamic programming algorithm working with a complexity polynomial in N and K (this is hence a pseudo-polynomial algorithm as N is encoding in binary, contrary to K).