Probabilistic Aspects of Computer Science: TD7
Reachability Objectives in MDP

Emile Contal*
http://econtal.perso.math.cnrs.fr/teaching

October 31, 2013

We are interested here in computing the minimum and maximum probabilities to reach a subset of states of a given MDP, and in describing policies achieving these optima.

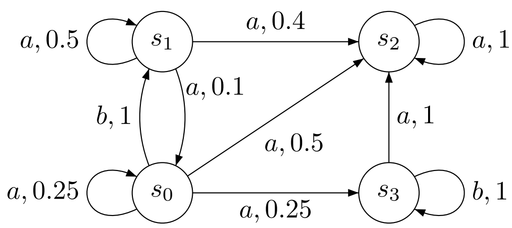
In the following, we consider an MDP $\mathcal{M} = (S, \{A_s\}_{s\in S}, p)$ with no reward functions. We recall that a *history* σ in $\mathcal{M}$ is an infinite sequence $(s_0, a_0, s_1, a_1, s_2, \ldots)$, such that for all $i \geq 0$, $s_i \in S$, $a_i \in A_{s_i}$ and $p(s_{i+1} \mid s_i, a_i) > 0$. Given an initial state $s \in S$, we denote $\mathsf{Hist}(s)$ the set of histories starting in s. Then a policy $\pi \in \Pi^{HR}$ (history-dependent and randomized) permits to define discrete-time Markov chain $\mathcal{M}^\pi$ with set of states being the finite prefixes of histories in $\mathsf{Hist}(s)$.

Given a target subset T of S, we denote as $\mathsf{Hist}(s, T)$ the set of histories, starting from state s, and reaching at some moment a state of T, i.e., such that there exists n with $s_n \in T$. As the reachability property only depends on a finite prefix of the history, $\mathsf{Hist}(s, T)$ is a measurable subset of $\mathsf{Hist}(s)$, hence, the DTMC $\mathcal{M}^\pi$ defines its probability, denoted $\mathbf{Pr}^\pi(s, T)$. In the following, we study the two quantities

$$p_{\min}(s,T) = \inf_{\pi\in\Pi^{HR}} \mathbf{Pr}^\pi(s,T) \qquad \text{and} \qquad p_{\max}(s,T) = \sup_{\pi\in\Pi^{HR}} \mathbf{Pr}^\pi(s,T)\,.$$

We will suppose in the following that states of T are absorbing, i.e., for all $t \in T$, $A_t = \{\alpha_t\}$ with $p(t \mid t, \alpha_t) = 1$.

You are invited to use the example depicted below throughout the rest, with $s = s_0$ and $T = \{s_2\}$.



Exercise 1 (Qualitative analysis)**.** We start by considering the problem of determining states s for which $p_{\min}(s,T)$ or $p_{\max}(s,T)$ is zero or one: we denote these four possible sets of states as $S_T^{\min=0}$, $S_T^{\min=1}$, $S_T^{\max=0}$ and $S_T^{\max=1}$.

1. Find an iterative algorithm to compute the four sets.
2. What is the complexity of your algorithms?

Exercise 2 (Stationary deterministic policies are enough)**.** We consider known, thanks to the previous exercise, the sets $S_T^{\min=0}$ and $S_T^{\min=1}$, and denote as $S^?$ the set $S \setminus (S_T^{\min=0} \cup S_T^{\min=1})$. We define (E) as the equation of the variable $\mathbf{x} \in \mathbb{R}^S$:

$$x_s = \begin{cases} 1 & \text{if } s \in S_T^{\min=1} \\ 0 & \text{if } s \in S_T^{\min=0} \\ \min_{a\in A_s} \sum_{s'\in S} p(s' \mid s, a) x_{s'} & \text{if } s \in S^?\,. \end{cases}$$

*Taken from last year exercises by Benjamin Monmege.

1. Show that the vector $(p_{\min}(s,T))_{s\in S}$ is a solution of (E).

2. Consider a stationary deterministic policy d^∞. Find a simple equation (E') having as unique solution the vector $(\mathbf{Pr}^{d^\infty}(s,T))_{s\in S^?}$. *(Hint: to prove uniqueness, you may search for the classification of the states of the underlying DTMC with set of states S.)*

3. Prove that (E) has then a unique solution and that $p_{\min}(s,T)$ is indeed a minimum computable by $\min_{\pi\in\Pi^{SD}} \mathbf{Pr}^\pi(s,T)$ (since Π^{SD} is a finite set). This shows the existence of an optimal strategy.

Exercise 3 (Computing $p_{\min}(s,T)$ and an optimal policy)**.** We will study the three principal methods enabling the computation of $p_{\min}(s,T)$ and an optimal policy: value iteration, linear programming and policy iteration.

1. Write a value iteration algorithm to estimate the probability $p_{\min}(s,T)$ and an associated almost optimal policy. *(Recall: value iteration is based on the iteration of the operator F suggested in equation (E) of the previous exercise, such that $p_{\min}(s,T)$ is its unique fixed point.)*

2. Show that the vector $(p_{\min}(s,T))_{s\in S}$ is the unique solution of the following linear programming:

$$\text{Maximize} \sum_{s\in S} x_s \text{ subject to } \begin{cases} \forall s \in S_T^{\min=1} & x_s = 1 \\ \forall s \in S_T^{\min=0} & x_s = 0 \\ \forall s \in S^? \ \forall a \in A_s & x_s \le \sum_{s'\in S} p(s' \mid s,a) x_{s'} \end{cases}$$

3. Write a policy iteration algorithm to find the exact value of $p_{\min}(s,T)$ and an associated optimal policy.

Exercise 4. Extend the previous exercises to the case of $p_{\max}(s,T)$.

Exercise 5 (When stationary deterministic policies are not enough...)**.** Find objectives which are more complex than reachability such that the maximal probability for this objective is not anymore obtainable with a stationary deterministic policy. In particular, find an example where histories are necessary, and another where randomization is necessary.