Tree Automata and Applications: TD3
Alternating Word Automaton

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Definition 1. The set $\mathbb{B}(\mathcal{P})$ of positive propositional formulae on the propositional variables $\mathcal{P}$ is the smallest set containing $\top$,$\bot$,$\mathcal{P}$ and closed by $\wedge$, $\vee$.

Definition 2. An Alternating Word Automaton (AWA) $\mathcal{A}$ on $\mathcal{F}$ is given by $(\mathcal{Q}, \mathcal{F}, q_0, \mathcal{Q}_f, \delta)$ with $\mathcal{Q}$ a finite set a states, $q_0 \in \mathcal{Q}$, $\mathcal{Q}_f \subseteq \mathcal{Q}$ and $\delta : \mathcal{Q} \times \mathcal{F} \to \mathbb{B}(\mathcal{Q})$. A computation of $\mathcal{A}$ on a word w is a tree t labelled by $\mathcal{Q}$ such that:

- the root is labelled by q_0,
- if $w = \epsilon$ then t is reduced to a leaf,
- if $w = a \cdot w'$, $t_1, \dots, t_n$ are the children of the root of t, and $\delta(q_0, a) = \phi$, then for all i, t_i is an computation of $(\mathcal{Q}, \mathcal{F}, q_i, \mathcal{Q}_f, \delta)$ on w' and $\{q_1, \dots, q_n\} \models \phi$.

An computation is *accepting* if all the leafs are labelled by a final state. The language accepted by $\mathcal{A}$ is the set of words for which there exists an accepting computation.

Definition 3. A computation t of $\mathcal{A}$ on w is *reduced* if t is a leaf or if $w = a \cdot w'$, $t = q_0(t_1, \dots, t_n)$ t_i is a computation of $\mathcal{A}$ starting from q_i on w' and:

- if S is a strict subset of $\{q_1, \dots, q_n\}$, then $S \not\models \delta(q_0, a)$,
- $t_1, \dots, t_n$ are reduced computations on w',
- if u and v are two subtrees of t with the same depth and the same label at the root, then $u = v$.

Example exercise. Let $\mathcal{F}$ be the alphabet $\{0,1\}$ and $\mathcal{A} = (\mathcal{Q}, \mathcal{F}, q_0, \mathcal{Q}_f, \delta)$ the AWA such that $\mathcal{Q} = \{q_0, q_1, q_2, q_3, q_4, q_1', q_2'\}$, $Q_f = \{q_0, q_1, q_2, q_3, q_4\}$ and:

$$\begin{array}{lll} \delta = \{ & q_0 0 \to (q_0 \wedge q_1) \vee q_1' & q_0 1 \to q_0 \\ & q_1 0 \to q_2 & q_1 1 \to \top \\ & q_2 0 \to q_3 & q_2 1 \to q_3 \\ & q_3 0 \to q_4 & q_3 1 \to q_4 \\ & q_4 0 \to \top & q_4 1 \to \top \\ & q_1' 0 \to q_1' & q_1' 1 \to q_2' \\ & q_2' 0 \to q_2' & q_2' 1 \to q_1' \quad \} . \end{array}$$

Give an example of an accepting computation of $\mathcal{A}$ on $w = 00101$ and an example of a non accepting computation of $\mathcal{A}$ on w.

Exercise 1. Let $\mathcal{A}$ be an AWA and t a accepting computation of $\mathcal{A}$ on w. Prove that there exists a reduced accepting computation $t'$of $\mathcal{A}$ on w.

Exercise 2. Prove that for all AWA, we can compute in exponential time a non-deterministic automaton which accepts the same language.

Exercise 3. Prove that for all AWA accepting a language L, we can compute in exponential time a deterministic automaton which accepts the set of mirror words of L.

Exercise 4. Give an AWA with size polynomial in n accepting the language $\{wxw^R \mid w \in (a+b)^n, x \in (a+b)^*\}$, where w^R is the mirror of w.

Bonus question : give a deterministic automaton with size exponential in n for this language.

Exercise 5. Given a non-deterministic bottom-up tree automaton $\mathcal{A}_T = (\mathcal{Q}, \mathcal{Q}_f, \mathcal{F}, \Delta)$, let $\mathcal{A}_W$ be the AWA on the singleton alphabet $\{1\}$ with states $\mathcal{Q} \times \mathcal{F}$ and transitions:

$$\delta\big((q,f),1\big) = \bigvee_{f(q_1,\ldots,q_n)\to q\in\Delta} \bigwedge_{i=1}^{n} \bigvee_{f_j\in\mathcal{F}} (q_i, f_j) \ .$$

Give the set of initial/final states of $\mathcal{A}_W$ such that $\mathcal{L}(\mathcal{A}_T) \neq \emptyset$ iff $\mathcal{L}(\mathcal{A}_W) \neq \emptyset$.

Show that if the formulae in $\mathbb{B}(\mathcal{Q})$ for an AWA $\mathcal{A}$ on a singleton alphabet are in DNF, we can reduce the emptiness problem of $\mathcal{A}$ to the emptiness problem for a tree automaton of polynomial size. That is, the emptiness problem for AWA on singleton alphabet is in PTIME when the formulae are in DNF.