Probabilistic Aspects of Computer Science: TD3
Random walks and applications

Emile Contal
http://econtal.perso.math.cnrs.fr/teaching

September 30, 2014

A random walk on an undirected graph is a special type of Markov chain that is often used in analyzing algorithms. Notions or results present in Exercise 1 will be used in Exercises 2, 3 and 5.

Exercise 1 (Cover time)**.** Let $G = (V, E)$ be a finite, undirected, and connected graph. A random walk on G is a Markov chain defined by the sequence of moves of a particle between vertices of G. In this process, the place of the particle at a given time step is the state of the system. If the particle is at vertex i and if i has $d(i)$ outgoing edges, then the probability that the particle follows the edge $\{i, j\}$ and moves to a neighbor j is $1/d(i)$.

1. Show that a random walk on an undirected graph G is aperiodic if and only if G is not bipartite.

2. In the rest of the exercise, we assume that G is not bipartite. Show that a random walk on G converges to a steady-state distribution π, where $\pi_v = \frac{d(v)}{2|E|}$.

3. We denote $\mu_{v,u}$ the expected number of steps to reach u from v. Show that if $\{u, v\} \in E$, then $\mu_{u,v} + \mu_{v,u} \leq 2|E|$.

4. The *cover time* of G is defined as the maximum over all vertices $v \in V$ of the expected time to visit all of the nodes in the graph by a random walk starting from v. Show that the cover time of G is bounded above by $2|E|(|V| - 1)$.

5. As an application, suppose we are given an undirected graph $G = (V, E)$ and two vertices s and t in G, and we want to determine whether there is a path connecting s and t. For simplicity, assume that the graph G has no bipartite connected components. By standard deterministic search algorithms, we can easily solve the problem in linear time, using $\Omega(n)$ space. Show that the following algorithm returns the correct answer with probability 1/2, and it only errs by returning that there is no path from s to t when there is such a path. What is the time and space complexities of this algorithm? (*Hint: you may use the Markov's inequality, which says for a random variable X and $a > 0$ that* $\mathbf{Pr}(|X| \geq a) \leq \frac{\mathbf{E}(|X|)}{a}$.)

> ***s*-*t* Connectivity algorithm**
> 1. Start a random walk from s.
> 2. If the walk reached t within $2|V|^3$ steps, return that there is a path. Otherwise, return that there is no path.

Exercise 2 (3-coloring)**.** A coloring of a graph is an assignment of a color to each of its vertices. A graph is k-colorable if there is a coloring of the graph with k colors such that no two adjacent vertices have the same color. Let G be a 3-colorable graph.

1. Show that there exists a coloring of the graph with two colors such that no triangle is monochromatic.

2. Consider the following algorithm for coloring the vertices of G with two colors so that no triangle is monochromatic (we will call that an almost 2-coloring in the sequel). The algorithm begins with an arbitrary coloring of G with 2 colors. While there are any monochromatic triangles in

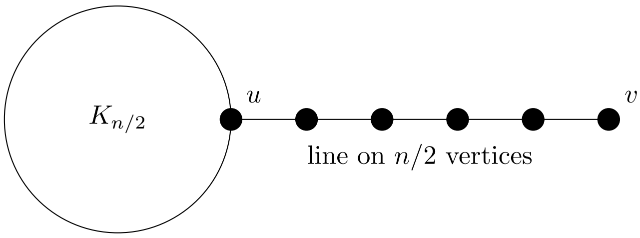
G, the algorithm chooses one such triangle and changes the color of a randomly chosen vertex of that triangle. Derive an upper bound on the expected number of such recoloring steps before the algorithm finds an almost 2-coloring of G.

Exercise 3 (Cat and mouse)**.** A cat and a mouse each independently take a random walk on a connected, undirected, non-bipartite graph G, with n vertices and m edges. They start at the same time on different nodes, and each makes one transition at each time step. The cat eats the mouse if they are ever at the same node at some time step. Show an upper bound of $O((nm)^2)$ on the expected time before the cat eats the mouse. What is a good strategy for the cat to eat quickly the mouse?

Exercise 4 (Computation of cover times)**.**

1. What is the cover time of a line when we start on an end node of the line?
2. What is the cover time of a complete graph?

The lollipop graph on n vertices is a clique on $n/2$ vertices connected with a line on $n/2$ vertices as shown below:



The node u is a part of both the clique and the line. Let v denote the other end of the line.

3. Show that the cover time of a random walk starting at v is $\Theta(n^2)$.
4. Show that the cover time of a random walk starting at u is $\Theta(n^3)$.

Exercise 5. For the following random walks, give the classification of the states (transient, null recurrent, or positive recurrent) and tell whether they admit a steady-state distribution.

1. the random walk over $\mathbb{Z}$?
2. the random walk on the 2-dimensional integer lattice, where each point has four neighbors (up, down, left, and right)?
3. the random walk on the 3-dimensional integer lattice?